Chapter 4 - Linear Functions

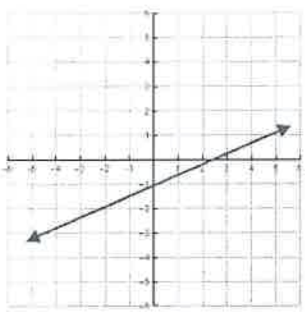
4.1 - Slope

The slope (gradient) of a linear equation describes the steepness and direction of a line. As we read graphs from left to right, we observe the vertical change relative to the horizontal progression (In other words: does the line travel up or down as you move from left to right).

The slope of a line tells you the RATE OF CHANGE and has the symbol m.

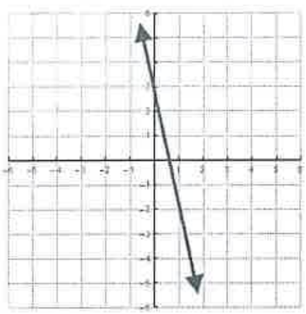
There are only four possible slopes a straight line can have:

Positive Slope



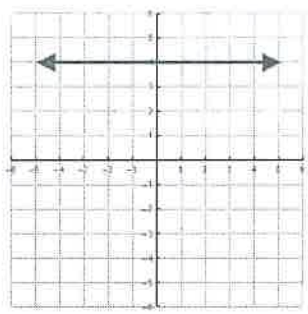
The line goes "up hill" as you go from left to right.

Negative Slope



The line goes "down hill" as you go from left to right.

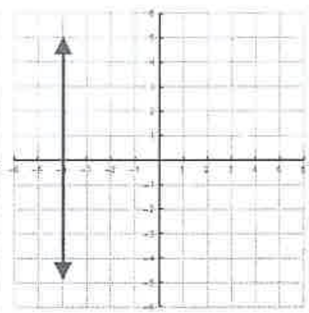
Zero Slope



The line is horizontal.

$$m = \frac{rise = 0}{run}$$

Undefined Slope



The line is vertical.

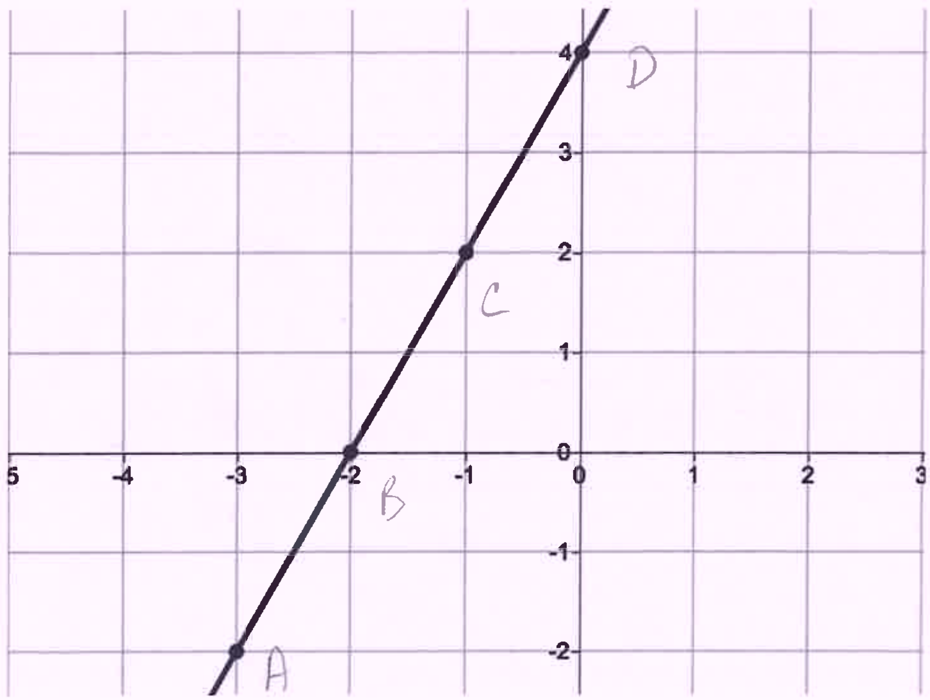
$$m = \frac{rise}{run = 0}$$

Finding Slope from a Graph

Slope requires the use of the following formula. We will come up with alternate ways of viewing and utilizing this equation, but they will always be based on the simple fact that:

$$Slope = m = \frac{vertical\ change}{horizontal\ change} = \frac{RISE}{RUN}$$

Take the following line



We can see that it intersects the points
A: (-3, -2), B: (-2, 0), C:(-1, 2), D:(0, 4)

*Our line obviously crosses through rational numbers between those distinct points like (-1.5, 1), but by choosing integers for our points will make the math easier.

To find the slope, pick ANY two points that the line crosses and observe the vertical displacement, and divide it by the horizontal displacement.

$$\text{Slope of segment } AB = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{up } 2}{\text{right } 1} = \frac{+2}{+1} = 2$$

$$\text{Slope of segment } BD = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{up } 4}{\text{right } 2} = \frac{+4}{+2} = 2$$

<u>Finding the Slope from Ordered Pairs</u>

While it is useful to be able to observe a graph and find the slope, a more powerful tool is being able to determine the slope based on a set of ordered pairs. It is the same principle as before, but now written out:

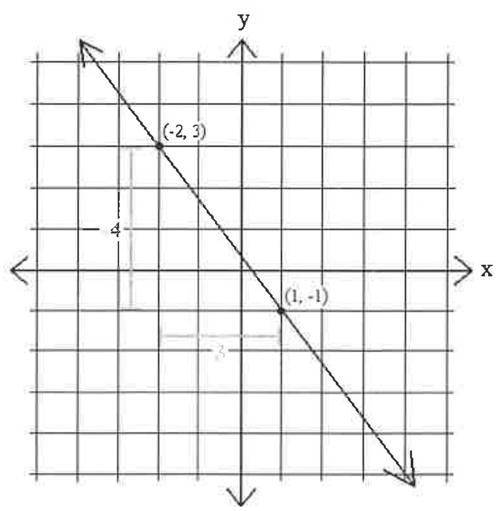
$$slope = m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

How to use this formula:

1) Find two points on your graph, write them out as ordered pairs.
2) Label the first set PAIR 1, and the second set PAIR 2
   a) The first pairing is thus (x_1, y_1) and the second is (x_2, y_2)
   b) It is recommended that you label your points with this notation!
3) Plug in the points into your formula, and solve.

Eg.

1) Pick two points: (-2, 3) and (1,-1)



2) Pair 1: (-2,3) Pair 2: (1,-1)
   (x_1, y_1) (x_2, y_2)

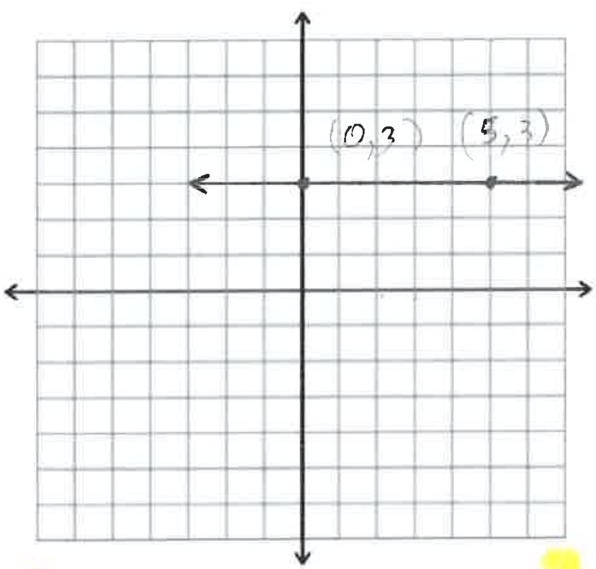
3) $\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1)-(3)}{(1)-(-2)} = \frac{-4}{3} = m$

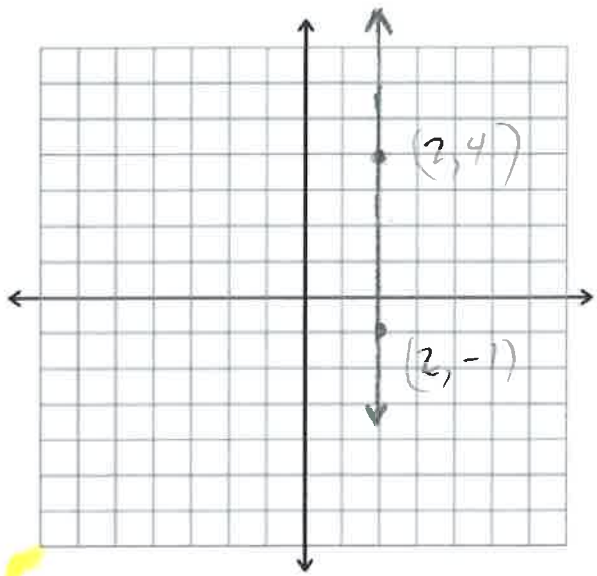
Lines with Zero Slope and Undefined Slope

We have now seen slopes that are positive and negative (which are the most common) but importantly we must discuss slopes of lines that are horizontal (which have zero slope) or vertical (undefined slope).

The best way to think of this is to imagine you are skiing:
How fast are you going when the hill is completely flat? 0
How fast are you going if the hill was completely vertical? ∞ *or* ∅

Now let's observe this mathematically:





$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3)-(3)}{(5)-(0)} = \frac{0}{5} = 0$$

1: (0,3)
2: (5,3)

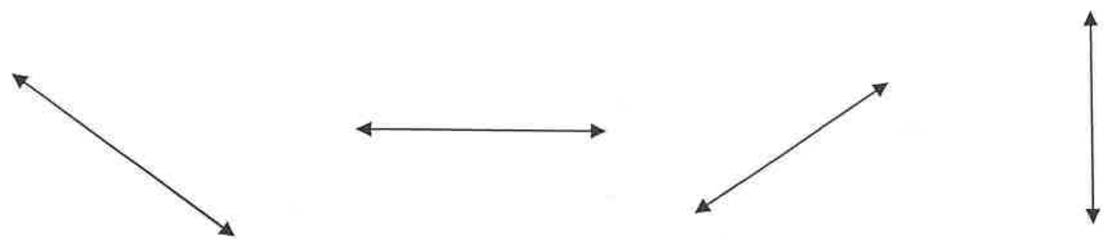
y=3

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4)-(-1)}{(2)-(2)} = \frac{5}{0} = \varnothing$$

1: (2,-1)
2: (2,4)

x=2

FMPC 10 4.1 The Slope of a Line Name: ____________________



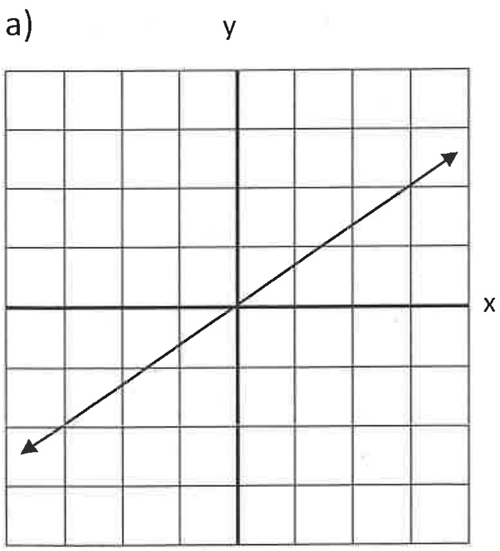
Negative Slope Zero Slope Positive Slope Undefined Slope

$$\textbf{SLOPE} = \frac{RISE}{RUN}$$

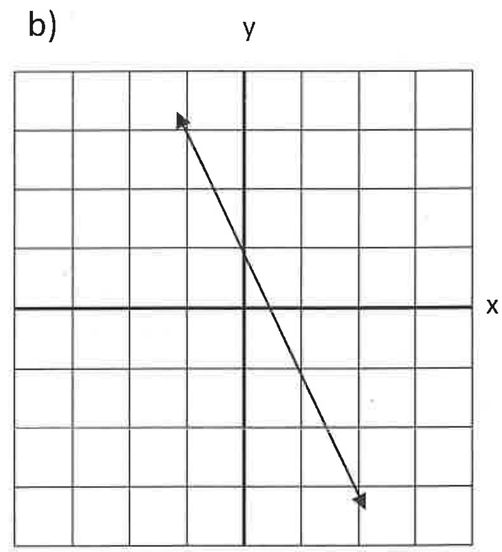
Indicate the rise and run on each of the graphs below.

What is the slope of each line?

a)



b)



Slope = ____________________

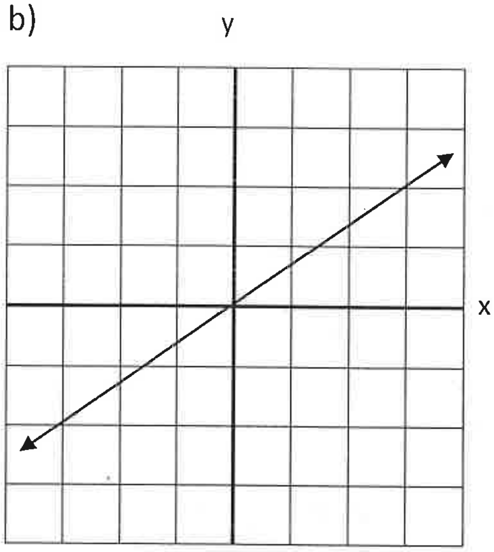
Slope = ____________________

Using the formula where slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Determine the slope of the line using the following steps:

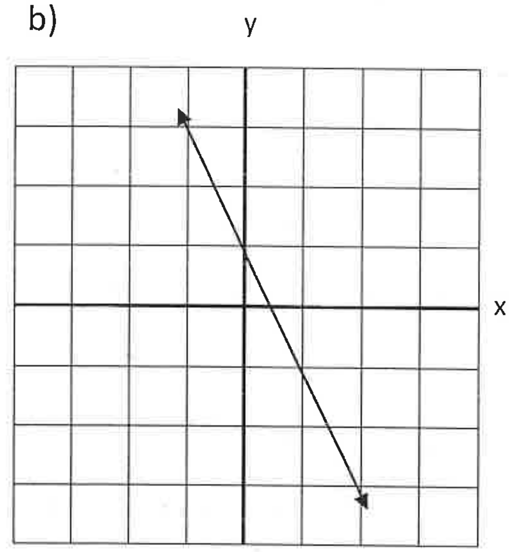
1. Pick a point on the line that intersects a point on the grid and label it (x_1, y_1)
2. Pick another point on the line and label it (x_2, y_2)
3. Apply the formula to determine the slope.

b)



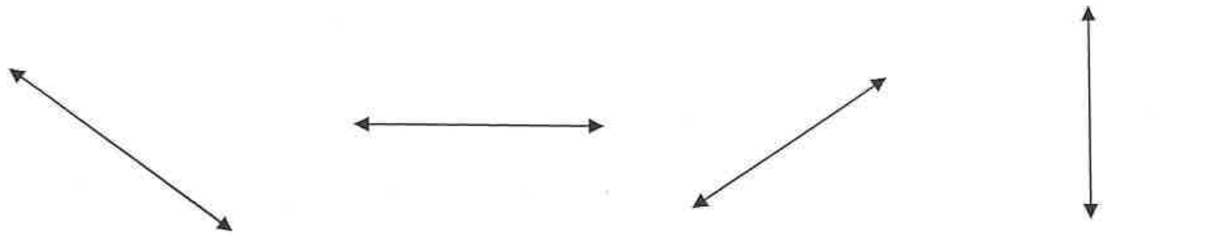
Slope = ____________________

b)



Slope = ____________________

FMPC 10 4.1 The Slope of a Line Name: Key



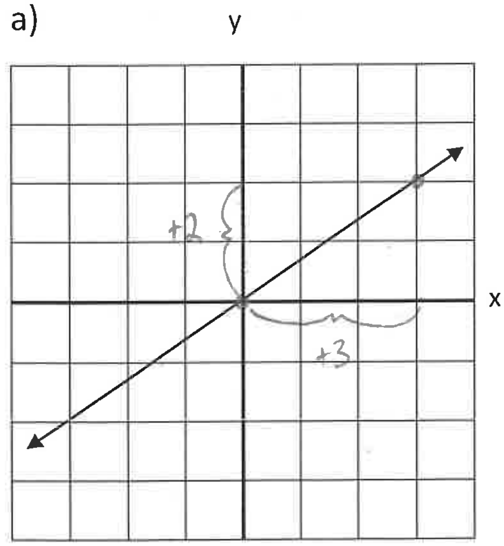
Negative Slope Zero Slope Positive Slope Undefined Slope

$$\textbf{SLOPE} = \frac{RISE}{RUN}$$

Indicate the rise and run on each of the graphs below.

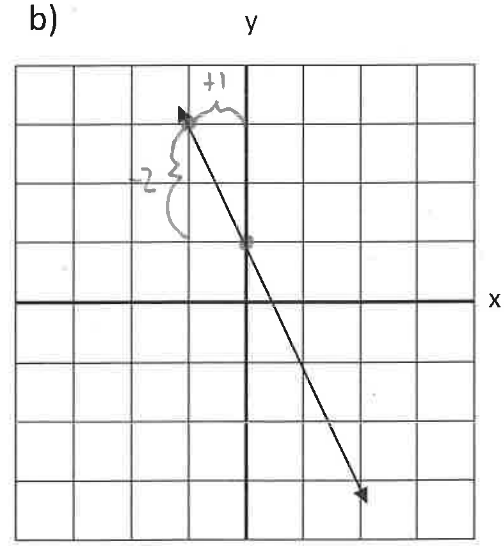
What is the slope of each line?

a)



Slope = $\frac{\uparrow 2}{\rightarrow 3} = \frac{+2}{+3}$ $\boxed{2/3}$

b)



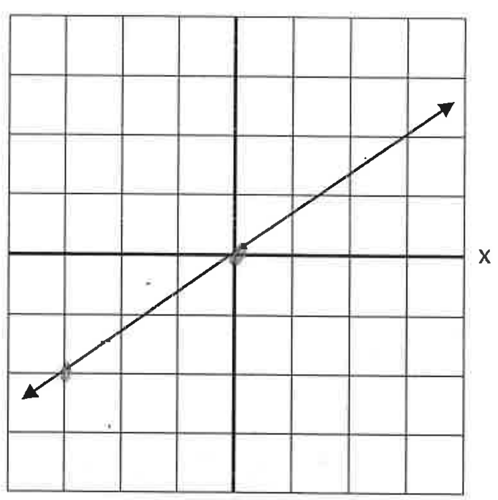
Slope = $\frac{\downarrow 2}{\rightarrow 1} = \frac{-2}{+1} = \boxed{-2}$

Using the formula where slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Determine the slope of the line using the following steps:

1. Pick a point on the line that intersects a point on the grid and label it (x_1, y_1)
2. Pick another point on the line and label it (x_2, y_2)
3. Apply the formula to determine the slope.

b)



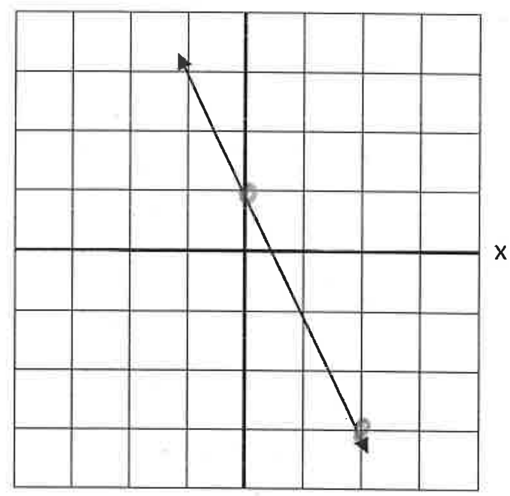
$(-3, -2)$ $(0, 0)$
x_1 y_1 x_2 y_2

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{0-(-2)}{0-(-3)} = \frac{+2}{+3}$

Slope = $2/3$

b)



$(0, 1)$ $(2, -3)$
x_1 y_1 x_2 y_2

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-3)-(1)}{(2)-(0)}$

$= \frac{-4}{2} = -2$

Slope = -2

4.2 - Rate of Change

Rates are measures of comparison between two different units. We see them all the time:

Kilometers per hour: ***km/h***

Dollars per hour: ***$/h***

Litres per 100km:***l/100km***

To be able to represent rates, we first must know what the amount of change that is taking place between any two actions. The term change in math is symbolized by the Greek letter "Delta" (Δ) . We can now use it in our formula from the previous section:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

This idea of change in one value over the change in another (or $\frac{\Delta y}{\Delta x}$) is fundamental to our understanding of a variety of concepts. For some of these, a simple calculation will do, but often for more continual functions, a graph is the best way to express the concepts at hand.

Example 1) If Parker works 40 hours a week, and his bi-weekly pay-cheque is $2400; how much does he make per hour?

Solution:

Bi-weekly = every two weeks = 80 hours.

$/*h*

$2400/80 hours = $30/*h*

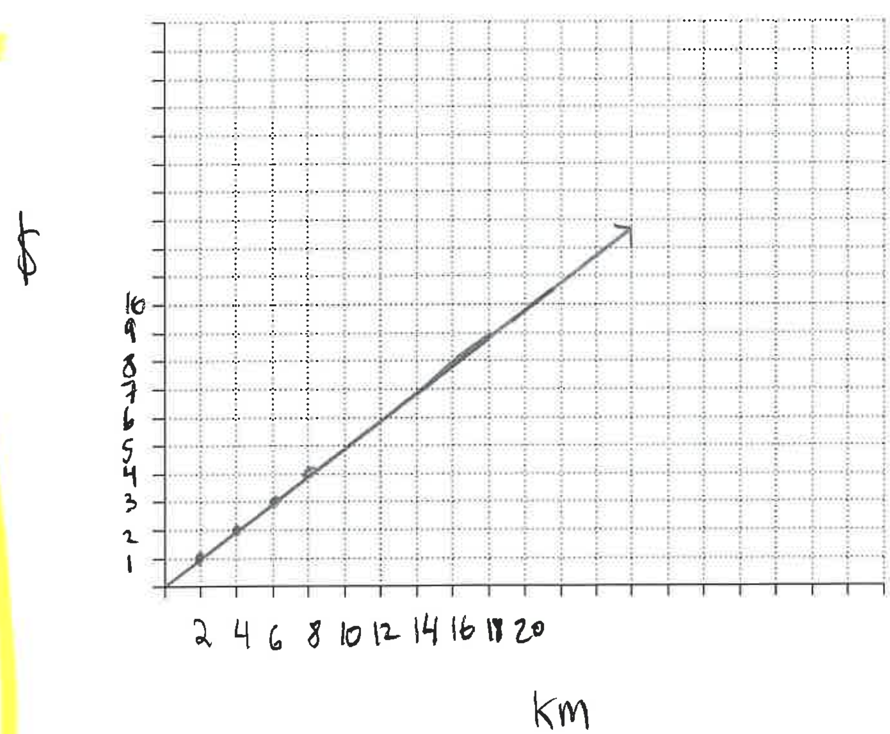
Example 2) At 1pm, Gemma rented a bicycle to travel around Elk Lake. She returned the bicycle at 5pm the same day after travelling 48km. It costs $24 to rent the bicycle.

a) Determine Gemma's average speed in km/hr
b) Determine the rental rate in dollars per hour
c) Determine the rental rate in dollars per km
d) Graph part (c)

a) $\frac{km}{hr} = \frac{48km}{5-1\,hr} = \frac{48}{4} = \boxed{12\ km/hr}$

b) $\frac{\$}{hr} = \frac{\$24}{4hr} = \boxed{\$6/hr}$

c) $\frac{\$}{km} = \frac{\$24}{48km} = \frac{\$1}{2km} = \boxed{\$.50/km}$



Example 3) The average single-family home in Victoria in 2009 cost ~$430,000; in 2018 the average price was ~$700,000. Assuming continuous linear growth:

a) Write a formula for a home's cost '*C*,' as a function of time *t*.

b) Draw a graph of this linear function

c) Determine the average home's value in 2014 and 2019

d) In what year were homes valued at $500,000? When will they be worth $1,000,000?

Solution:

a) $y = mx + b$

$C = m(t) + b$

$\frac{\Delta y}{\Delta x} = \frac{700000 - 430000}{2018 - 2009}$

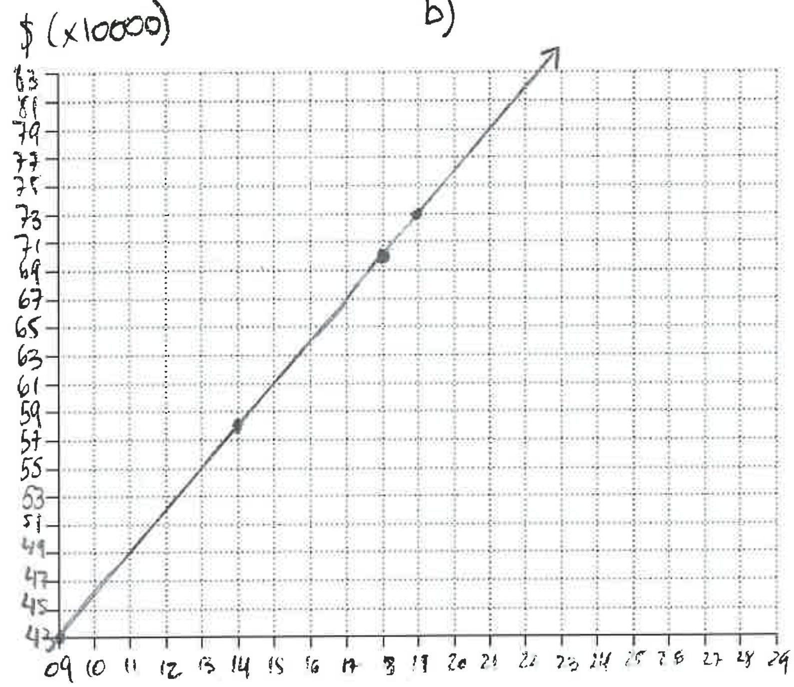
$= \frac{270000}{9} = \frac{\$30000}{yr}$

$C = 3000t + 430000$

↓

$C = 30000(t - 2009) + 430000$

b)



c) From Graph: 2014 → $580,000

2019 → $730,000

d) $500000 → ~2011

$10000000 is off graph so:

$1000000 = 30000(t - 2009) + 430000$

$570000 = 30000(t - 2009)$

$19 = t - 2009$

$t = 2028$

4.3 - Graphing Linear Functions

To graph a line, there are only two main pieces that we need:

1) The Slope $(m) = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
2) Any point (most often the y-intercept (the b in the formula $y=mx+b$)

<u>Intercepts</u>

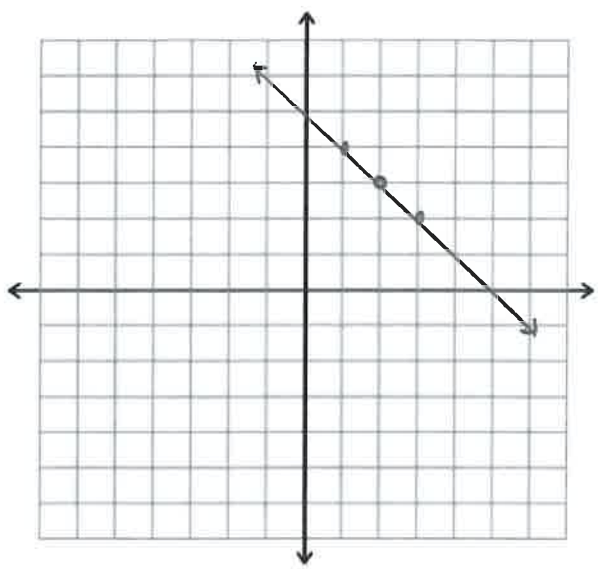
The point at which a function crosses an axis is known as an intercept.

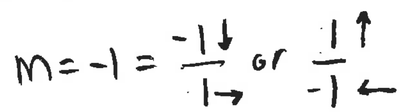
When the graph crosses the x-axis (which you can observe has a y-value of 0), it is known as the x-intercept and has the coordinate of $(x,0)$. When the graph crosses the y-axis (which you can observe has an x-value of 0), it is known as the y-intercept and has the coordinate of $(0, y)$.

These points are integral to your understanding of graphing and are your go-to points when first trying to draw a function on a graph.

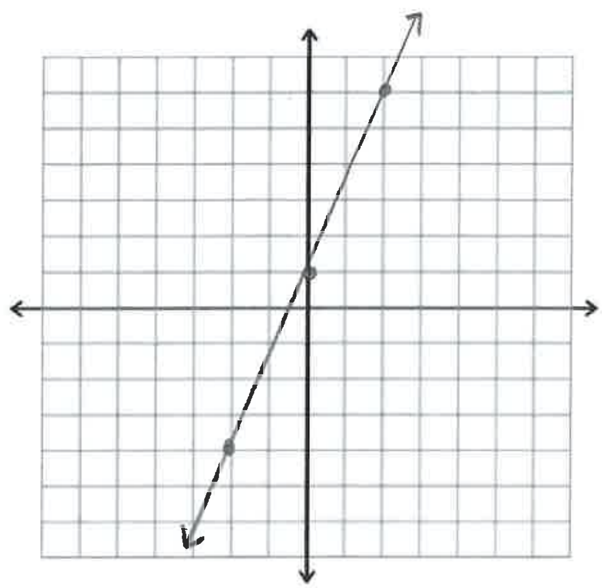
In this section, we will graph lines when given a variety of different qualifiers.

Example 1) Graph a line with slope -1, going through the point (2, 3)





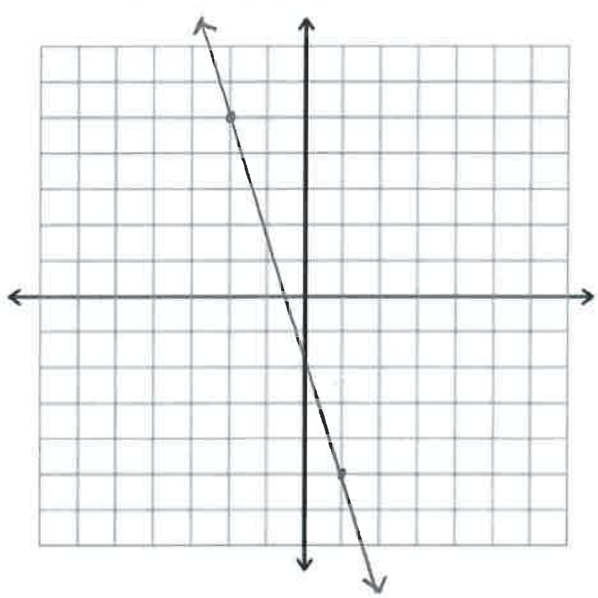
Example 2) Graph the line and determine a point in quadrant III for a line with slope 2.5 going through the point (0, 1).



$m = 2.5 = \frac{5\uparrow}{2\rightarrow}$

QIII : (-2, -4)

Example 3) Determine the slope of the graph:

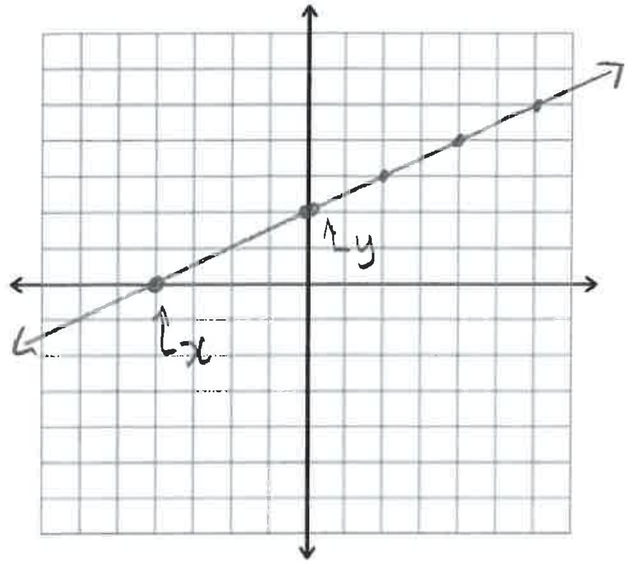


Point 1: (-2, 5)

Point 2: (1, -5)

$\frac{\Delta y}{\Delta x} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-5}{1-(-2)} = \boxed{\frac{-10}{3}} = m$

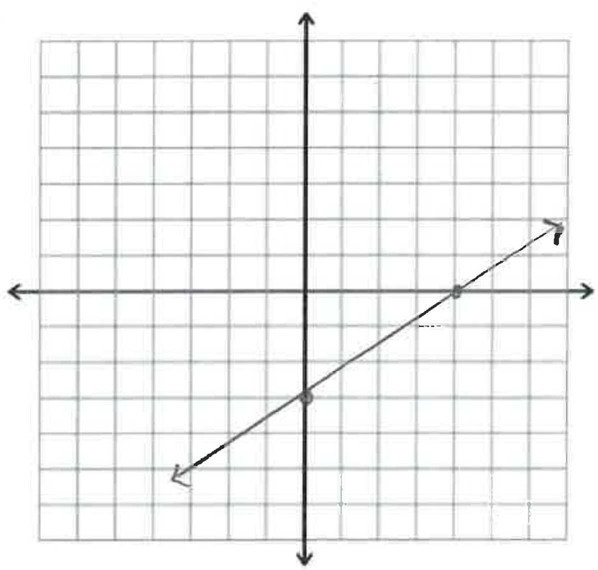
Example 4) Graph the line and determine the x-intercept and y-intercept of the line with slope ½, going through the point (4, 4).



x-int: (-4, 0)

y-int: (0, 2)

Example 5) Determine the slope, and graph the line with an x-intercept at 4 and a y-intercept at -3.



x-int: $(4, 0)$ (x_1, y_1)

y-int: $(0, -3)$ (x_2, y_2)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-3)-(0)}{(0)-(4)}$

$= \frac{-3}{-4} = \boxed{\frac{3}{4} = m}$

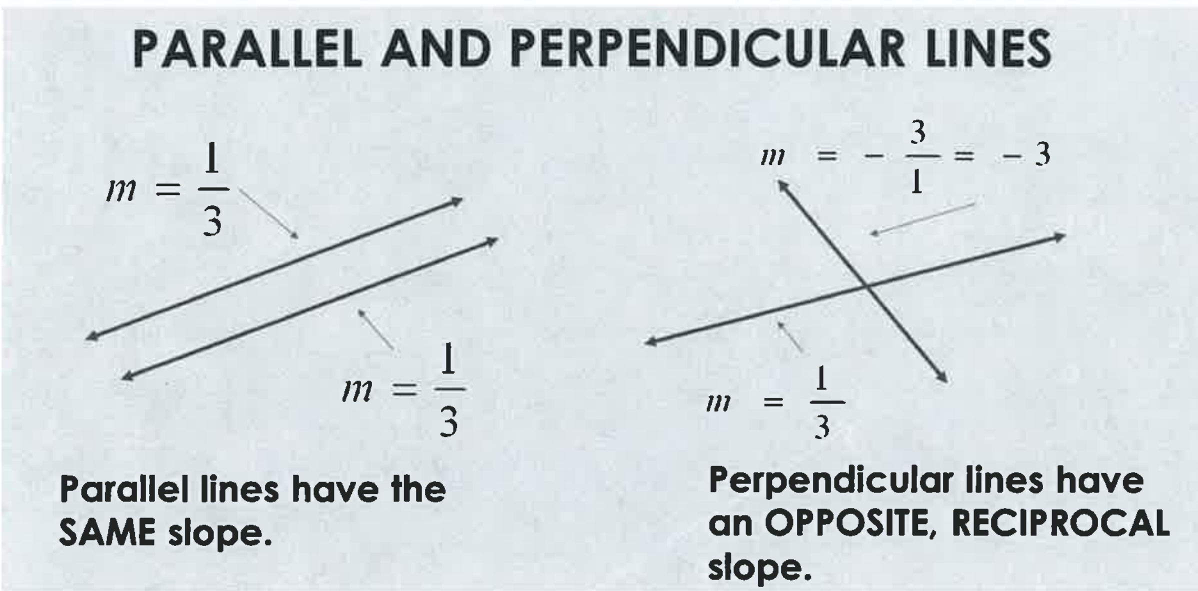
4.4 - Parallel and Perpendicular Lines

Parallel Lines: are lines in a coordinate system that never intersect (never meet or cross). The only way this can happen is if the lines have identical slopes (the rise or fall at the exact same rate!). If the slope of line one (m_1) is $\frac{a}{b}$ than the slope of line two (m_2) is also $\frac{a}{b}$. $m_1 = m_2$

Perpendicular Lines: are lines that form right angles when they intersect (the easiest visual for this is the intersection of *x* and *y-axes*). If $m_1 = \frac{a}{b}$ than $m_2 = \frac{-b}{a}$. This relationship between the two lines is known as the "negative reciprocal." If we multiply m_1 *and* m_2 we would get -1. $m_2 = -\frac{1}{m_1}$

Eg. $\frac{a}{b} \times \frac{-b}{a} = -\frac{ab}{ba} = -\frac{ab}{ab} = -1$

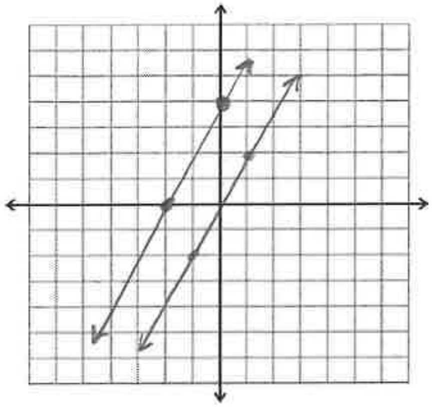
To find the negative reciprocal of a fraction, simply flip the fraction and change the negativity.



Examples: Determine if the line through the first pair of points is parallel to, perpendicular to, or neither compared to the second pair of points.

a) (-1, -2) and (1, 2); (-2,0) and (0,4)
b) (-4, 2) and (0, 3); (-3, -2) and (3,2)
c) (0, -4) and (-1, -7); (3, 0) and (-3, 2)
d) (-*x*, -4) and (4, *x*); (*x*, 3) and (3, *x*)

a)



1: $(-1,-2)$ and 2: $(1,2)$ $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{1-(-1)}$

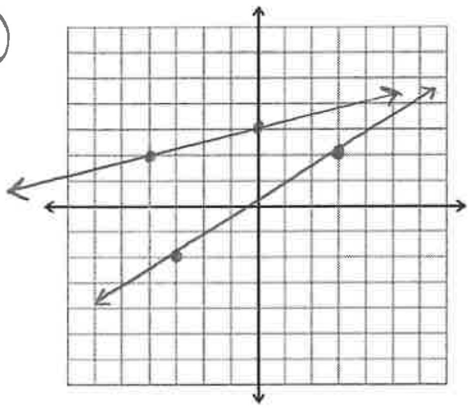
$= 4/2 = 2/1 = \boxed{2 = m_1}$

1: $(-2,0)$ and 2: $(0,4)$

$= \frac{\Delta y}{\Delta x} = \frac{4-0}{0-(-2)} = \frac{4}{+2} = \frac{2}{1} = \boxed{2 = m_2}$

$\boxed{m_1 = m_2 = \text{parallel}}$

b)

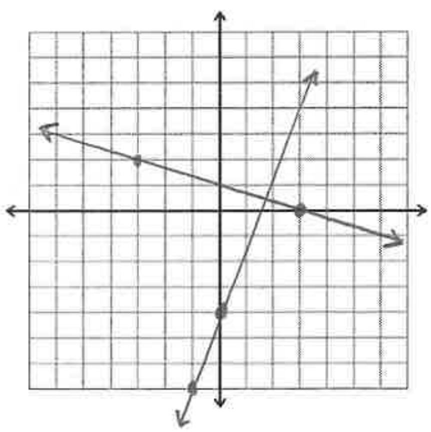


1: $(-4,2)$ and 2: $(0,3)$ $= \frac{\Delta y}{\Delta x} = \frac{3-2}{0-(-4)} = \frac{1}{4}$

1: $(-3,-2)$ and $(3,2)$ $= \frac{\Delta y}{\Delta x} = \frac{2-(-2)}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$

m_1 unrelated to m_2 $\therefore$ $\boxed{\text{Not} \parallel \text{or} \perp}$

c)



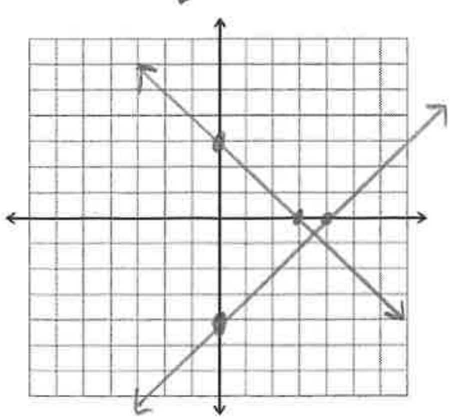
1: $(0,-4)$ and 2: $(-1,-7)$ $= \frac{\Delta y}{\Delta x} = \frac{-7-(-4)}{-1-0} = \frac{-3}{-1} = \frac{3}{1}$

1 and 2

$(3,0)$ $(-3,2)$ $= \frac{\Delta y}{\Delta x} = \frac{2-0}{-3-3} = \frac{2}{-6} = \frac{-1}{3}$

$m_1 = -\frac{1}{m_2}$ $\therefore$ $\boxed{m_1 \perp m_2}$

d)



Let $x = 0$

1: $(-x,-4)$ 2: $(4,x)$ $= \frac{\Delta y}{\Delta x} = \frac{x - -4}{4 - -x} = \frac{x+4}{4+x}$

$= \frac{x+4}{x+4} = 1$

1: $(x,3)$ 2: $(3,x)$ $= \frac{\Delta y}{\Delta x} = \frac{x-3}{3-x} = \frac{x-3}{-x+3}$

$= \frac{x-3}{-(x-3)} = \frac{-1}{1} = -1$

$\therefore$ $\boxed{m_1 \perp m_2}$

4.5 - Applications of Linear Relations

Example 1)

After graduating from school you go out on a job hunt, receiving two job offers!

Job 1 offers you \$20/hour with a \$100 bonus per week.

Job 2 offers you \$15/hour with a \$275 bonus per week.

Which job should you take if you plan on working:

a) 30 hours a week

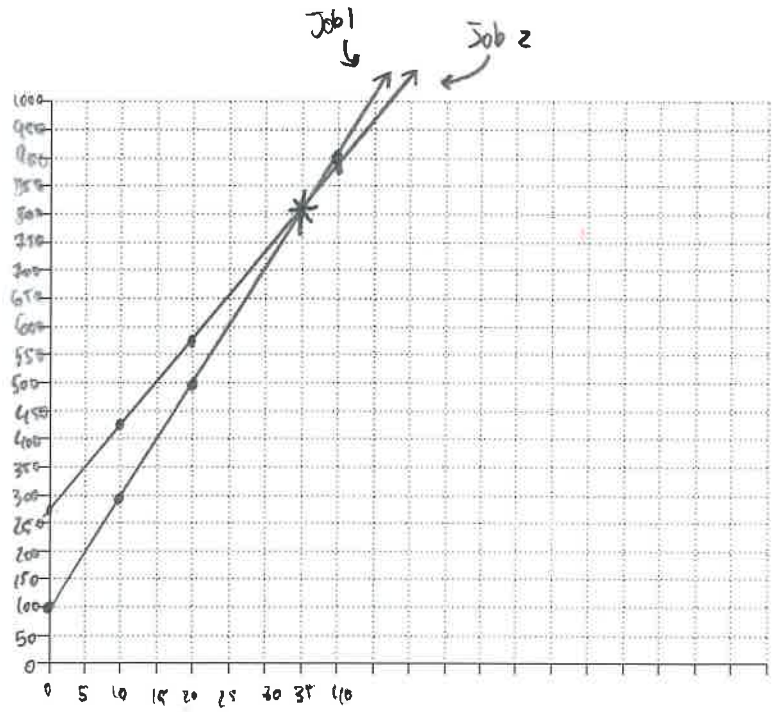
b) 40 hours a week

c) At how many hours per week do the two jobs pay the same?

Write a formula for each job and graph your solutions to answer questions a, b, c.

$J_1 = 20h + 100$

$J_2 = 15h + 275$



a) $J_1 = 20(30) + 100$

$= 600 + 100 = \$700$

* $J_2 = 15(30) + 275$

$= 450 + 275 = \$725$

b) * $J_1 = 20(40) + 100$

$= 800 + 100 = \$900$

$J_2 = 15(40) + 275 =$

$= 600 + 275 = \$875$

c) $J_1 = J_2$

$20h + 100 = 15h + 275$

$5h = 175$

$h = 35$

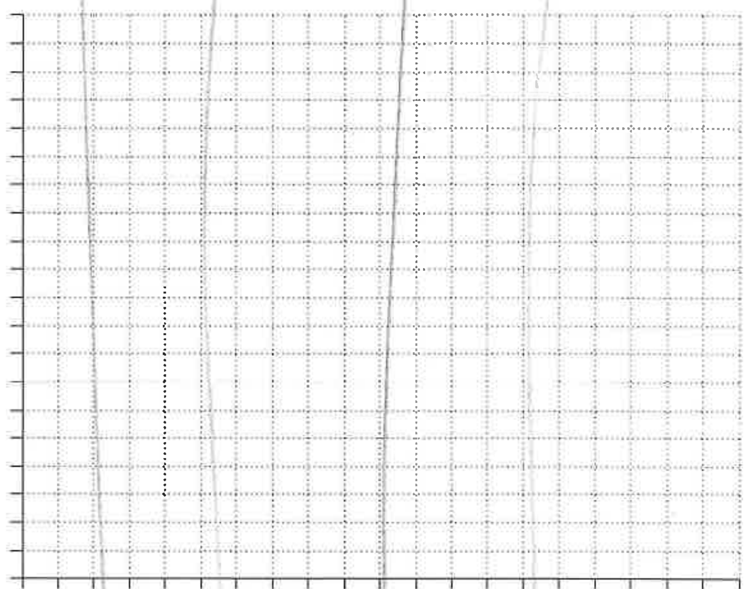
Example 2) When you take a taxi, there is an initial charge of $3.25, plus an extra $1.88 per km (for the sake of mental math we will round this up to $2/km).

a) Write the equation that shows the cost as a function of distance.

b) Graph your equation.

c) What price do you pay for going 13km?

d) How far did you travel if you paid $40 for a cab ride?



Notes: